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Nonperturbative tests of the parent/orbifold correspondence in supersymmetric gauge theories

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Abstract: It has been shown that a procedure analogous to orbifolding in string theory, when applied to certain large-N field theories, leaves correlators invariant perturbatively. We test nonperturbative agreement of some aspects of the parent and orbifold theories. More specifically, we find that the holomorphic gauge couplings on the Coulomb branch in the parent and orbifold Seiberg-Witten theories are related, even away from the 't Hooft limit. We also check that any large-N theory which has an infrared conformal fixed point and satisfies certain anomaly positivity constraints required by such theories will continue to satisfy those constraints after orbifolding. We discuss extensions of these results to finite N.

Keywords: Nonperturbative Effects, Duality in Gauge Field Theories.

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Contents

1. Introduction

Motivated by the AdS/CFT correspondence [1], and the preservation of this correspondence upon orbifolding of the supergravity theory [2], a relation between field theories and their orbifolds was derived in [3]. It was shown that at large N all correlators in the orbifolded theory are simply related in perturbation theory to the same correlators in the parent theory. It need not be the case that the parent field theory is derived from string theory.

In section 2 we review orbifolding in field theories, and in section 3 we discuss the perturbative result of Bershadsky and Johansen [3]. In section 4 we study nonperturbative extensions of this correspondence between field theories and their orbifolds. The result is that if a supersymmetric theory and its orbifold have Coulomb branches, the Seiberg-Witten period matrix of the orbifolded theory is simply related to that of the parent theory. This relation is valid for all N and coupling g. In section 5 we study the anomaly positivity constraints [4] on supersymmetric theories with infrared fixed points and find that they are satisfied in orbifolds of large-N theories with infrared fixed points. We study two classes of theories at finite N and find that the positivity conditions hold for orbifolds of these theories, as well. Conclusions are summarized in section 6.

2. Orbifolding in field theory

By orbifolding in field theory, we will mean removing from the theory all states which are not invariant under some discrete subgroup of the internal symmetry (gauge and global) of the theory. Unlike orbifolding in string theory, we do not orbifold space-time so we will look at quantum field theories in flat four dimensional Minkowski space. In cases where the four dimensional theory can be realized as a world-volume theory on D-branes which are part of some brane configuration, orbifolding the space transverse to these branes

JHEP12(2002)047

corresponds to orbifolding the field theory which lives on the world volume of the branes except that we do not include the twisted sector fields in the field theory orbifold [5]. There are various restrictions on string theory orbifolds which follow from string theory consistency requirements such as tadpole cancelation. From the field theory point of view, the only restriction is that the orbifolded field theory does not have any gauge anomalies. We will now discuss some examples of orbifolding in field theory. We will always use the regular representation (see below) of the orbifold group G to embed it in the gauge group. If the field theory is realized as a world-volume theory in some brane configuration, such an embedding arises naturally. The importance of the regular representation is not clear from a purely field theoretic point of view. However, it simplifies the analysis in perturbation theory [3]. We can also embed G in the global symmetries of the theory. Different embeddings in the global symmetries will lead to different orbifold theories.

SU(kN) pure gauge theory orbifolded by $\mathbb{Z}_k$

As discussed above, $\mathbb{Z}_k$ is embedded by its regular representation in the gauge group. In general, a discrete group $G = \{g_1, g_2, \ldots, g_k\}$ has a regular representation given by $k \times k$ matrices γ^a defined by $g_a g_i = g_j (\gamma^a)_{ji}$. Using the fact that $g_a g_b \neq g_b$ unless $g_a = 1$ (1 is the identity element of the group which we will also denote by g_1), we get

$$\mathrm{Tr}\, \gamma^a = k \delta_1^a \, .$$

The regular representation is reducible and by a suitable choice of basis can be brought to a block diagonal form such that each irreducible representation R_i appears with multiplicity equal to its dimension $d_i = \dim(R_i)$ along the diagonal. This implies that $\sum_i d_i^2 = k$. For the group $\mathbb{Z}_k$, in an appropriate basis, the regular representation matrices are given by

$$\begin{aligned} \gamma^a &= \mathrm{diag}\{1, (\omega^a), (\omega^a)^2, \ldots, (\omega^a)^{k-1}\} \, , \qquad a \neq 1 \, , \\ \gamma^1 &= \mathrm{diag}\{1, 1, 1, \ldots, 1\} \, , \end{aligned}$$

where $\omega = e^{2\pi i/k}$ and $\omega^k = 1$. Now it is easy to embed the group $\mathbb{Z}_k$ in the gauge group SU(kN). The matrices,

$$\begin{aligned} \Gamma_N^a &= \mathrm{diag}\{\mathbf{1}, (\omega^a) \times \mathbf{1}_N, (\omega^a)^2 \times \mathbf{1}_N, \ldots, (\omega^a)^{k-1} \times \mathbf{1}_N\} \, , \qquad a \neq 1 \, , \\ \Gamma_N^1 &= \mathrm{diag}\{\mathbf{1}_N, \mathbf{1}_N, \mathbf{1}_N, \ldots, \mathbf{1}_N\} \end{aligned}$$

form a $\mathbb{Z}_k$ subgroup of SU(kN). This means that the action of the orbifold group on the gauge field matrix $A_\mu = A_\mu^a T^a$ is given by $A_\mu \to \Gamma_N^i A_\mu \Gamma_N^{i\dagger}$. The components left invariant by the orbifold group are then $N \times N$ blocks along the diagonals. Hence the gauge group of the orbifolded theory is $\mathrm{SU}(N)_1 \times \mathrm{SU}(N)_2 \times \cdots \times \mathrm{SU}(N)_k$. (Here we ignored anomalous gauge U(1)'s.)

SU(kN) theory with Adjoint scalars orbifolded by $\mathbb{Z}_k$

Now consider an SU(kN) gauge theory with complex scalars Φ in the adjoint representation of the gauge group. This theory has a U(1) global symmetry $\Phi \to e^{i\alpha}\Phi$. We can embed

	$\mathrm{SU}(N)_1$	$\mathrm{SU}(N)_2$	$\mathrm{SU}(N)_3$	$\cdots$	$\mathrm{SU}(N)_k$
Q_1	$\Box$	$\overline{\Box}$	1	$\cdots$	1
Q_2	1	$\Box$	$\overline{\Box}$	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Q_k	$\overline{\Box}$	1	1	$\cdots$	$\Box$

Table 1: Matter content of the orbifolded SU(kN) theory with adjoints.

$\mathbb{Z}_k$ non-trivially in this global U(1) group as $\{\omega^j, j = 0, \ldots, k-1\}$ where $\omega = e^{2\pi i/k}$. The invariant scalars are in $N \times N$ blocks shifted to the right of the diagonal. The matter content of the orbifolded transforms as in table 1.

$\mathcal{N} = 2$, SU(kN) pure gauge theory orbifolded by $\mathbb{Z}_k$

In $\mathcal{N} = 1$ language, the $\mathcal{N} = 2$ pure gauge theory has a vector superfield and a chiral superfield in the adjoint representation of the gauge group. This theory possesses a global U(1) symmetry under which the gauge field and its fermionic partner transform as $(A_\mu, \lambda) \to (A_\mu, \lambda)$, and the adjoint scalar and its fermionic partner as $(\phi, \psi) \to e^{i\alpha}(\phi, \psi)$. This symmetry is anomalous but there is a discrete non-anomalous subgroup Z_{2kN}, which in turn has a $\mathbb{Z}_k$ subgroup generated by $\omega = e^{2\pi i/k}$. We identify this $\mathbb{Z}_k$ with the orbifold group. The gauge group is embedded via the regular representation as usual. The orbifolded theory is an $\mathcal{N} = 1$ supersymmetric $\mathrm{SU}(N)^k$ theory with chiral multiplets transforming as in the table above. We will discuss the relation between the two theories in the Coulomb phase in section 4.

$\mathcal{N} = 1$, SU(kN) theory with kF flavors orbifolded by $\mathbb{Z}_k$

This theory has a $\mathrm{SU}(kF)_L \times \mathrm{SU}(kF)_R$ global symmetry and we use F-fold copies of the regular representation to embed the orbifold group $\mathbb{Z}_k$ in each factor of the flavor group. We first embed the orbifold group trivially in the other global symmetries (various U(1)'s). The orbifolded theory is $\mathcal{N} = 1$ supersymmetric $\mathrm{SU}(N)^k$ theory where each factor is disconnected and has F flavors. We can also choose to embed the orbifold non-trivially in the $\mathrm{U}(1)_R$ symmetry under which the gauginos and flavors have charge +1 (implying that the fermionic quarks are uncharged)(this theory is discussed in detail in [6]). This symmetry is anomalous but has a non-anomalous Z_{kN} subgroup which in turn has a $\mathbb{Z}_k$ subgroup. This we identify with the orbifold group. We will use Q,$\bar{Q}$, Ψ,$\bar{\Psi}$ for the scalar and fermionic components of the superfields transforming in the fundamental and anti-fundamental representation of the gauge group. Then, under the orbifold group, the fields will transform as $A_\mu \to \Gamma_N^a A_\mu \Gamma_N^{a\dagger}$, $\lambda \to \omega^{a-1}\Gamma_N^a \lambda \Gamma_N^{a\dagger}$, $Q \to \omega^{a-1}\Gamma_N^a Q \Gamma_F^{a\dagger}$, $\bar{Q} \to \omega^{*a-1}\Gamma_N^{a\dagger}\bar{Q}\Gamma_F^a$, $\Psi \to \Gamma_N^a \Psi \Gamma_F^{a\dagger}$, and $\bar{\Psi} \to \Gamma_N^{a\dagger}\bar{\Psi}\Gamma_F^a$, where $\omega = e^{2\pi i/k}$. The orbifolded theory has no supersymmetry and the following matter content, and matter content as in table 2 (for $k = 3$).

Orbifolding $\Box\!\Box$ and $\begin{smallmatrix}\Box\\[-2pt]\Box\end{smallmatrix}$ under SU(kN) by $\mathbb{Z}_k$

The symmetric tensor may transform under some global U(1) as $\Box\!\Box$ ($\begin{smallmatrix}\Box\\[-2pt]\Box\end{smallmatrix}$) $\to e^{i\alpha}\Box\!\Box$ ($\begin{smallmatrix}\Box\\[-2pt]\Box\end{smallmatrix}$). If we embed the orbifold group trivially in this U(1), $\Box\!\Box$ ($\begin{smallmatrix}\Box\\[-2pt]\Box\end{smallmatrix}$) under SU($kN$) will become

	SU(N)	SU(N)	SU(N)	SU(F)$_L$	SU(F)$_L$	SU(F)$_L$	SU(F)$_R$	SU(F)$_R$	SU(F)$_R$
λ_1	$\square$	$\overline{\square}$	1	1	1	1	1	1	1
λ_2	1	$\square$	$\overline{\square}$	1	1	1	1	1	1
λ_3	$\overline{\square}$	1	$\square$	1	1	1	1	1	1
Q_1	$\square$	1	1	1	$\overline{\square}$	1	1	1	1
Q_2	1	$\square$	1	1	1	$\overline{\square}$	1	1	1
Q_3	1	1	$\square$	$\overline{\square}$	1	1	1	1	1
$\bar{Q}_1$	$\overline{\square}$	1	1	1	1	1	1	1	$\square$
$\bar{Q}_2$	1	$\overline{\square}$	1	1	1	1	$\square$	1	1
$\bar{Q}_3$	1	1	$\overline{\square}$	1	1	1	1	$\square$	1
Ψ_1	$\square$	1	1	$\overline{\square}$	1	1	1	1	1
Ψ_1	1	$\square$	1	1	$\overline{\square}$	1	1	1	1
Ψ_1	1	1	$\square$	1	1	$\overline{\square}$	1	1	1
$\bar{\Psi}_1$	$\overline{\square}$	1	1	1	1	1	$\square$	1	1
$\bar{\Psi}_2$	1	$\overline{\square}$	1	1	1	1	1	$\square$	1
$\bar{\Psi}_3$	1	1	$\overline{\square}$	1	1	1	1	1	$\square$

Table 2: Matter content of the SU($3N$) theory with $3F$ flavors orbifolded by $\mathbb{Z}_3$.

$\square\!\square$ ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$) under each of the factors of SU$(N)^k$. However, if we choose to embed it non-trivially as $\{\omega^j,\ j = 0,\ldots,k-1\}$, we get bifundamentals under the k factors if k is odd and bifundamentals under $k-1$ factors and $\square\!\square$ ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$) under one of the factors if k is even.

3. Perturbative correspondence between parent and orbifolded theories

In this section, we review the arguments by Bershadsky and Johansen [3] who proved that at large N, all correlators of the orbifolded theory are identical to the corresponding correlators in the parent theory upto a rescaling of the coupling constants. In the orbifolding procedure, the coupling constants $g_{\rm orb}$ of the various factors of the product groups are the same as that of the parent theory. However, if we define $g^2_{\rm orb} = k g^2_{\rm parent}$, the correlators of the two theories are identical.

We define a projector onto states which are invariant under the orbifold group by

$$P = \frac{1}{k}\sum_{a=1}^{k} \mathcal{R}^a\,, \tag{3.1}$$

where $\mathcal{R}^a$ are matrices in a particular (generally reducible) representation of the orbifold group. $P^2 = P$ and $P = 1$ in the trivial representation, and $P = 0$ in all other irreducible representations. Thus when acting on some field in representation R, the projector projects the invariant components. For example, for adjoint fields, the projector can be defined as

$$P = \frac{1}{k}\sum_{a=1}^{k} r_a \otimes \Gamma_a^\dagger \otimes \Gamma_a \tag{3.2}$$

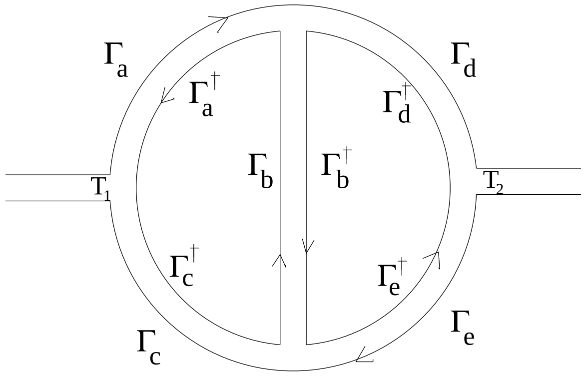

Figure 1: A two loop planar Feynman diagram

where r_a is the action of the $\mathbb{Z}_k$ subgroup of the global symmetry, $\Gamma_a^\dagger$ acts on the anti-fundamental index and Γ_a acts on the fundamental index. At large N, the perturbation series is dominated by planar Feynman diagrams with arbitrary number of loops [7]. Each diagram factorizes into a product of certain kinematic (group theory) factor and another factor with complicated momentum and spin dependence which is independent of the internal symmetry structure of the diagram.

Consider the planar diagram shown in figure 1. The group theory factor is

$$g^4 \frac{1}{k^5} \sum_{a,b,c,d,e=1}^{k} \mathrm{Tr}\left[T_1 \Gamma_a \Gamma_d T_2 \Gamma_e \Gamma_c\right] \mathrm{Tr}\left[\Gamma_c^\dagger \Gamma_b \Gamma_a^\dagger\right] \mathrm{Tr}\left[\Gamma_b^\dagger \Gamma_e^\dagger \Gamma_d^\dagger\right] \tag{3.3}$$

For simplicity, we are assuming that the orbifold group is embedded trivially in the global symmetry group. Since the matrices Γ are in the regular representation, the diagram is zero unless

$$\Gamma_c^\dagger \Gamma_b^\dagger \Gamma_a^\dagger = 1$$

and

$$\Gamma_b \Gamma_e^\dagger \Gamma_d^\dagger = 1\,.$$

Then (3.3) becomes

$$g^4 N^2\, \mathrm{Tr}(T_1 T_2)\,.$$

The same diagram in the parent theory will be proportional to

$$g^4 N^2 k^2\, \mathrm{Tr}(T_1 T_2)\,.$$

The factor of $(Nk)^2$ comes from summing over Nk particles running in the two loops. The momentum and spin dependence of the same diagram in both theories are identical so we see that the two diagrams are the same upto rescaling of couplings. The general proof is given in [3] and [6].

4. Orbifolds of Seiberg-Witten theories

As discussed in the last section it has been demonstrated that, at least perturbatively at large N, all correlators of the orbifolded theory are equivalent to the corresponding correlators in the parent theory up to a rescaling of the gauge coupling at some fixed large scale. In this section we demonstrate that at least one aspect of the non-perturbative behavior of orbifolded and parent theories are related at large N, namely the gauge coupling functions of Seiberg-Witten theories. In the 't Hooft limit g^2N is held fixed with $g \to 0$. Then instanton corrections, which are proportional to powers of $e^{-8\pi^2/g^2}$, are generally negligible. Exceptions which involve Nth roots of instanton corrections, for example the gaugino condensate [6], are non-vanishing in the 't Hooft limit and vary as $e^{-\frac{8\pi^2}{g^2N}}$. Furthermore, as discussed in [8], monopoles which become massless at large N lead to important nonperturbative effects. We find a simple relation between the (inverse) gauge coupling functions of the parent and orbifolded theories nonperturbatively for all N and g.

For simplicity we study the case where the curves for the orbifolded and parent theories are hyperelliptic. Then the gauge coupling function at generic points in moduli space, where the gauge group is broken to a product of U(1) factors, is given by the period matrix of the corresponding curve and is easily expressed in terms of integrals over cycles of the curve. The hyperelliptic curve can be written in the form

$$y^2 = f_{2r+2}(x, s_i), \tag{4.1}$$

where the subscript $2r+2$ is the order of the polynomial f in x and r is the genus of the curve, which for pure $\mathcal{N}=2$ supersymmetric Yang-Mills theory is equal to the rank of the corresponding gauge group. The moduli s_i $(i = 1, \ldots, r)$ which parameterize the Coulomb branch are the vacuum expectation values of the symmetric gauge invariant operators of the theory.

In [5] a prescription was given for generating the curves for $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(N)^k$ gauge theory with bifundamental chiral multiplets from those of the $\mathcal{N}=2$ $\mathrm{SU}(kN)$ pure gauge theory by orbifolding the $\mathcal{N}=2$ theory by the abelian discrete group $\mathbb{Z}_k$. The curve of the orbifolded theory, which was obtained by other means in [9], was obtained by keeping only those terms of the parent theory with moduli invariant under the $\mathbb{Z}_k$, and rescaling $x^k \to x$ in the resulting curve.

We demonstrate here how the period matrix of the curve corresponding to the orbifolded theory is related to that of the parent theory. To be specific we will consider orbifolding of an $\mathrm{SU}(kN)$ gauge theory by $\mathbb{Z}_k$. The genus r curve (4.1) ($r = kN-1$ here) has $2r$ cycles which are divided into a cycles and b cycles with symplectic intersection $a_i \cdot b_j = \delta_{ij}$ and $a_i \cdot a_j = b_i \cdot b_j = 0$. In our case the cuts on the x-plane will have a $\mathbb{Z}_k$ symmetry, and we label cycles schematically as follows. We choose a non-symplectic basis of cycles (α_i, β_i) as in figure 2. Then the appropriate symplectic basis is $a_i = \sum_{k=1}^{i} \alpha_k$, $b_i = \beta_i$. The subset of cycles relevant for comparison with the orbifolded theory are shown in figure 3. They correspond to the $\mathbb{Z}_k$ invariant a cycles, as discussed below.

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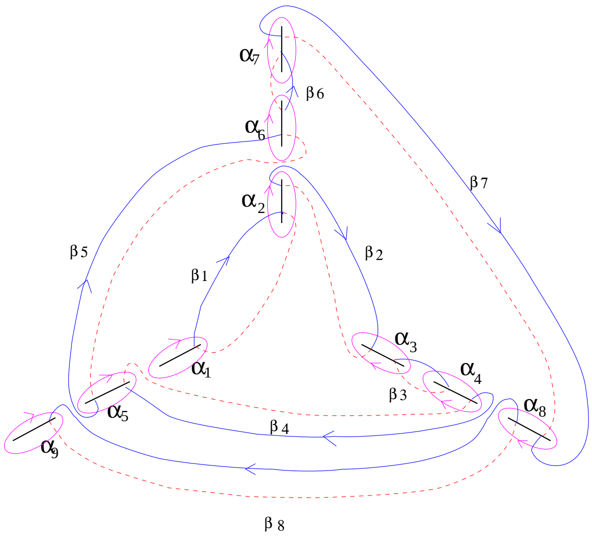

Figure 2: A non-symplectic basis of cycles. On the orbifold sector of moduli space the cut x plane is symmetric under the orbifold group, in this case Z_3.

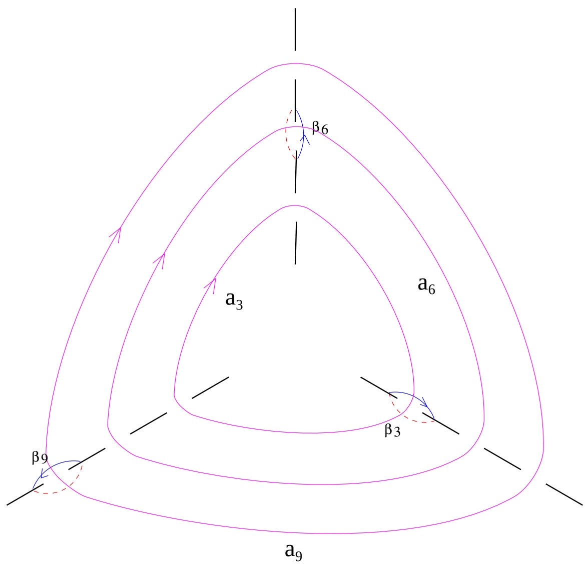

Figure 3: The orbifold sector of a symplectic basis of cycles.

The genus r hyperelliptic curve also has a basis of r holomorphic differentials which can be written

$$\omega_j = \frac{x^{j-1}\,dx}{y(x)} \qquad j = 1, \ldots, r\,. \tag{4.2}$$

The matrices of a periods and b periods of the curve are given by integrals of the differentials (4.2) over the a and b cycles, respectively,

$$A_{ij} = \int_{a_i} \omega_j\,, \qquad B_{ij} = \int_{b_i} \omega_j\,. \tag{4.3}$$

The period matrix of the curve (4.1) is then given by

$$\tau_{jl} = B_{jk} A^{-1}_{kl} . \tag{4.4}$$

The identification of the period matrix τ_{jl} of the hyperelliptic curve with the gauge coupling function in the $\mathcal{N} = 2$ SU(kN) theory is made via [10, 11, 12]

$$\tau_{jl} = \frac{\partial \mathbf{a}^D_j}{\partial \mathbf{a}_l} , \tag{4.5}$$

where $\mathbf{a}_l$ is the vacuum expectation value of the diagonalized adjoint scalars and $\mathbf{a}^D_l$ are their duals as described in [10]. If for some curve the period matrix (4.4) satisfies appropriate monodromies around the singularities of the curve, then comparing (4.4) and (4.5) it is natural to set

$$\frac{\partial \mathbf{a}^D_j}{\partial s_k} = \int_{b_j} \omega_k , \qquad \frac{\partial \mathbf{a}_j}{\partial s_k} = \int_{a_j} \omega_k . \tag{4.6}$$

Then the orbifold invariant sector of the adjoint VEV's $\mathbf{a}_j$ corresponds to the invariant sector of a cycles and differentials. Alternatively, the VEV's are given directly by integrals of the Seiberg-Witten differential [8] $\lambda = (1/2\pi i)(x/y)dP(x)$ where $P(x)$ is the polynomial that appears in the curve $y^2 = P(x)^2 - \Lambda^{2N}$ of the $\mathcal{N} = 2$ pure gauge theory. Since the Seiberg-Witten differential is not invariant under the $\mathbb{Z}_k$ symmetry, only the integrals over the $\mathbb{Z}_k$ invariant cycles will be invariant. Hence, the invariant a_j correspond to the invariant a cycles. As matrices, we reorganize the periods in a convenient way: The first $(N-1) \times (N-1)$ block of A_{jl} corresponds to the orbifold invariant sector, which we will continue to call $A_{km,kn}$.

In order to compare corresponding points in the moduli space of the orbifolded and parent theories, we set all non-invariant moduli in the parent theory to zero. We then study that sector of the gauge coupling functions of the parent theory that correspond to the U(1) gauge group factors that survive on the moduli space of the orbifolded theory. In the basis of cycles described above, these factors correspond to τ_{jl} for j and l equal to multiples of k, where k is the order of the orbifold group $\mathbb{Z}_k$. It is claimed that this sector of the inverse period matrix of the curve corresponding to the parent theory is related to the period matrix of the orbifolded theory. This is verified as follows.

We can think of the curve $y(x)$ as being defined on a double sheeted cover of the cut x-plane with branch points at the roots of $y(x)$ connected pairwise to form branch cuts. On the orbifold sector of the moduli space of the parent theory the curve is a function only of x^k, and the roots are of the form $x^k = p_i$. The roots can then be labeled by p_i and a k-th root of unity. The branch cuts and cycles on the x-plane can be distributed as in figures 2 and 3. The roots of the orbifolded curve are then given by p_i. The holomorphic differentials (4.2) corresponding to the orbifolded sector are of the form

$$\omega_{kj} = \frac{x^{kj-1}\, dx}{y(x^k)} , \qquad j = 1, \ldots , r . \tag{4.7}$$

These differentials are invariant under multiplication of x by $e^{2m\pi i/k}$, so the integrals over the α and β cycles in figure 2 are invariant under similar rotations. The noninvariant

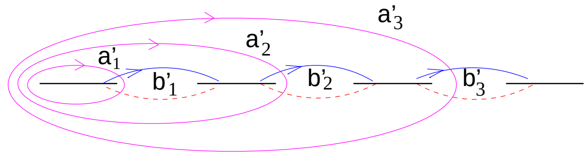

Figure 4: Cycles of the orbifolded theory.

differentials are all multiplied by $e^{2m\pi i/k}$ for some integer m under multiplication of x by $e^{2\pi i/k}$. Then the integrals of the noninvariant differentials over the invariant a cycles vanish. For example, suppose $\omega \to \omega e^{2m\pi i/k}$ when $x \to x\, e^{2\pi i/k}$. Then

$$\int_{a_k} \omega_{\text{noninv}} = \left(\sum_{n=0}^{k-1} e^{\frac{2mn\pi i}{k}}\right) \qquad \int_{\alpha_k} \omega_{\text{noninv}} = 0\,. \tag{4.8}$$

Now, consider the invariant sector of the matrix A_{jl} of a periods. It is given by

$$\begin{aligned} A_{kj,kl} &= \int_{a_{kj}} \frac{x^{kl-1}dx}{y(x^k)} = \int_{a_{kj}} \frac{\tilde{x}^{l-1}d\tilde{x}}{k\,y(\tilde{x})} \\ &= \int_{a'_j} \frac{\tilde{x}^{l-1}d\tilde{x}}{y(\tilde{x})} \\ &= A^{\text{orb}}_{jl}\,, \end{aligned} \tag{4.9}$$

where in the first line of (4.9) we changed variables $x \to \tilde{x} = x^k$, and the cycles a'_j are the a cycles of the orbifolded theory, drawn schematically in figure 4. An additional factor of k arises in the second line because the invariant a cycles become a sum over the a cycles of the orbifolded theory k times. Hence, the orbifold invariant sector of the a periods of the parent theory is equivalent to the matrix of a periods of the orbifolded theory.

Hence, the matrix A_{jl} takes the block lower triangular form

$$A = \begin{pmatrix} \begin{pmatrix} \\ A^{\text{orb}} \\ \\ \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} \\ \int_{a_{\text{non}}} \omega_{\text{inv}} \\ \\ \end{pmatrix} & \begin{pmatrix} \\ \int_{a_{\text{non}}} \omega_{\text{noninv}} \\ \\ \end{pmatrix} \end{pmatrix}. \tag{4.10}$$

A similar analysis applies to the b periods. By invariant b cycles we will mean those b cycles that have intersection with the invariant a cycles, and similarly for noninvariant b cycles. In this case, the integrals of the invariant differentials over the noninvariant cycles vanish. The reason is that the integrals are all of the form

$$\int_{p_i^{1/k}}^{\omega p_i^{1/k}} \frac{x^{kj-1}\,dx}{y(x^k)}\,, \tag{4.11}$$

for some root p_i of $y(x)$. Then letting $x \to x^k$ the path of integration contracts to a point, and the integral (4.11) vanishes.

In the basis of cycles and one forms given above, the invariant sector of b periods of the parent theory is simply rescaled compared to the b periods of the orbifolded theory,

$$
\begin{aligned}
B_{kj,kl} &= \int_{b_{kj}} \frac{x^{kl-1}dx}{y(x^k)} = \int_{b_{kj}} \frac{\tilde{x}^{l-1}d\tilde{x}}{k\,y(\tilde{x})} \\
&= \int_{b'_j} \frac{\tilde{x}^{l-1}d\tilde{x}}{k\,y(\tilde{x})} \\
&= \frac{1}{k} B^{\mathrm{orb}}_{jl} .
\end{aligned} \tag{4.12}
$$

There is no additional factor of k in the second line of (4.12) as in (4.9) since the relevant b cycles of the parent theory corresponds to single b cycles of the orbifolded theory and not a multiple cover as for the a cycles. Hence, the matrix of b periods takes the block upper triangular form

$$
B = \begin{pmatrix} \begin{pmatrix} \frac{1}{k} B^{\mathrm{orb}} \end{pmatrix} & \begin{pmatrix} \int_{b_{\mathrm{inv}}} \omega_{\mathrm{noninv}} \end{pmatrix} \\ \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} \int_{b_{\mathrm{non}}} \omega_{\mathrm{noninv}} \end{pmatrix} \end{pmatrix} . \tag{4.13}
$$

The orbifold sector of the inverse (or dual) period matrix, $\tau^{-1} = AB^{-1}$, of the parent theory is then simply related to that of the orbifolded theory. Namely,

$$
\tau^{-1\,\mathrm{parent}}_{kj,kl} = k\tau^{-1\,\mathrm{orb}}_{jl} . \tag{4.14}
$$

Therefore, the period matrix of the orbifolded theory is determined by that of the parent theory.

This is the main result of this section. Equation (4.14) relates the gauge coupling functions of the orbifold theory and the orbifold sector of the parent theory. It is valid for all N and g, and can be extended to more complicated orbifolds.

5. Anomaly positivity tests

In a series of papers [4, 13] it was demonstrated that in supersymmetric theories at conformal fixed points there are constraints on global anomalies which follow from unitarity. The argument relies on the supersymmetry multiplet structure which mixes the R-charge anomaly and the stress tensor trace anomaly, and positivity of central functions which appear in the operator product expansion of a product of two currents. At conformal fixed points these central functions are central charges which can be calculated in terms of global anomalies by 't Hooft anomaly matching. This procedure is not valid away from a fixed point because corrections to the relation between the central functions and anomalies are proportional to the β function [4]. The argument can be reversed to give evidence for or against the existence of a conformal fixed point in a theory. One assumes that a theory has a fixed point and calculates the various central charges in terms of 't Hooft matched global anomalies. If a central charge calculated this way is negative, then the theory could not have a conformal fixed point. If the central charge is positive when calculated this way, then a strong statement cannot be made, but positivity provides weak evidence for the existence of a conformal fixed point.

The Weyl anomaly coefficient, which must be positive at an infrared fixed point, can be written in terms of the $\mathrm{U}(1)_R^3$ and $\mathrm{U}(1)_R$ anomalies as [4]

$$c_{\mathrm{IR}} = \frac{1}{32}(9\,U(1)_R^3 - 5U(1)_R) = \frac{1}{32}\left(4\dim G + \sum_i (\dim R_i)(1-r_i)\left(5 - 9(1-r_i)^2\right)\right), \tag{5.1}$$

where $\dim G$ is the dimension of the gauge group and the sum is over all representations R_i of matter chiral multiplets in $\mathcal{N}=1$ language with R-charges r_i. For example, for $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(N)$ QCD with N_f flavors of chiral multiplets, the 't Hooft matched R charge of each flavor is $\frac{N_f-N}{N_f}$. The Weyl anomaly coefficient is [4]

$$c_{\mathrm{IR}}^{\mathrm{SQCD}} = \frac{1}{32}\left(4\,(N^2-1) + 2NN_f\left(\frac{N}{N_f}\right)\left(5 - 9\left(\frac{N}{N_f}\right)^2\right)\right), \tag{5.2}$$

which is easily checked to be positive in the conformal region $3N/2 < N_f < 3N$.

If the theory has a global flavor symmetry, then the flavor central charge is required to be positive. At a conformal fixed point it is given by

$$b_{\mathrm{IR}} = -3\,\mathrm{U}(1)_R F^2 = 3\sum_{ij}(\dim R_i)(1-r_i)\mu_i\,, \tag{5.3}$$

where μ_i is the Dynkin index of the representation R_i.

The Euler anomaly coefficient a, which is believed to satisfy the Zamolodchikov C theorem in four dimensions [4, 14, 15], is also expected to be positive at fixed points. In addition, the C theorem requires that the flow of the Euler anomaly be positive: $a_{\mathrm{UV}} - a_{\mathrm{IR}} > 0$. The relations between the Euler anomaly and the R current anomalies are [4],

$$\begin{aligned} a_{\mathrm{IR}} &= \frac{3}{32}(3\,\mathrm{U}(1)_R^3 - \mathrm{U}(1)_R) = \frac{3}{32}\left(2\dim G + \sum_i (\dim R_i)(1-r_i)\left(1 - 3(1-r_i)^2\right)\right) \\ a_{\mathrm{UV}} - a_{\mathrm{IR}} &= \frac{1}{96}\sum_i (\dim R_i)(3r_i - 2)^2(5 - 3r_i)\,. \end{aligned} \tag{5.4}$$

Assume a theory that flows to a conformal fixed point in the infrared has a unique anomaly free R symmetry. Then the positivity conditions are satisfied in that theory. If we consider orbifolding that theory, then the positivity conditions will remain to be true at large N. This is the case because the dimension of the orbifolded group at large N is rescaled by $1/k$, as are the dimensions of each matter representation. The anomaly free R-charges remain the same in this procedure. Hence, c_{IR} is rescaled by $1/k$ in the orbifolded theory, but otherwise is the same as c_{IR} in the parent theory. Hence, positivity is preserved at large N. That this is true for all theories which are obtained as orbifolds of theories with

JHEP12(2002)047

conformal fixed points provides evidence that the orbifolded theories also have conformal fixed points, as expected by the large N orbifold correspondence. One should note that the anomaly positivity conditions of [4] rely on supersymmetry, so we only consider orbifolds to supersymmetric theories. The anomaly calculations do not rely on a planar diagram expansion, so this result is valid also away from the 't Hooft limit at large N.

In our canonical example, large N, $\mathcal{N}=2$ SU(kN) pure gauge theory orbifolded by $\mathbb{Z}_k$ to $\mathcal{N}=1$ SU$(N)^k$ with bifundamentals, the dimension of the parent gauge group is $(kN)^2$, whereas in the orbifolded theory it is N^2 for each SU(N) factor, or kN^2 total. Similarly, there are k bifundamental chiral multiplets in the orbifolded theory, as opposed to one adjoint chiral multiplet in the parent theory. The dimension of each of the k bifundamentals is N^2, so again the dimension of the representation is rescaled by $1/k$ in the orbifolded theory compared with the parent theory. The anomaly free R charges of the adjoint chiral multiplet in the parent theory and the bifundamental chiral multiplets in the orbifolded theory are 0.

At finite N the anomalies are not simply rescaled by $1/k$, so the above discussion of preservation of the positivity conditions does not carry through. This might be related to the problem of additional U(1)'s that appear in the orbifolded theory which decouple at large N, but not otherwise. If we naively add one U(1) gaugino for each SU(N) or SU(kN) factor, then the anomalies would be simply rescaled by $1/k$ as for large N. Although positivity in orbifolded theories is not guaranteed as it is for large N, we have surveyed orbifolds of a few theories with duals in the conformal regime and have not found violation of any of the positivity conditions in any of those orbifolds. A theory which were to violate positivity would imply a violation of the orbifold correspondence at finite N, since then orbifolds of certain theories with infrared fixed points would not have infrared fixed points and correlators of these theories in the infrared would not match.

In [4] sufficient conditions on the R charges r_i of the chiral multiplets in an $\mathcal{N}=1$ theory were given for the various positivity conditions to be met. Since orbifolding does not change the R charges, if a parent theory satisfies these sufficient conditions then so will the orbifolded theory. The result of [4] was that in all renormalizable models studied there the flow $a_{\rm UV}-a_{\rm IR}$ satisfied the sufficient condition for positivity $r_i \leq \sqrt{5}/3$ for all chiral superfields ϕ_i; in all models not requiring an accidental U(1) symmetry for unitarity, $b_{\rm IR}$ and $c_{\rm IR}$ were positive by virtue of $1-\sqrt{5}/3 < r_i < 1$ for all ϕ_i. Hence we only need to check positivity for $a_{\rm IR}$, and for $b_{\rm IR}$ and $c_{\rm IR}$ in theories with accidental U(1) symmetry. What follows is two simple tests that we have done. It would be useful to continue this program by testing the positivity constraints for orbifolds of other theories.

In the conformal region of $\mathcal{N}=1$ supersymmetric SU(kN) gauge theory with kN_f flavors, $3N_c/2 < N_f < 3N_c$, the anomaly positivity conditions were shown to be satisfied in [4]. If the theory is orbifolded by embedding the orbifold group trivially in the U$(1)_R$, then the orbifolded theory is described by k copies of $\mathcal{N}=1$ SU(N) gauge theory with N_f flavors. This theory is also in the conformal regime and satisfies the anomaly positivity conditions.

The Kutasov-Schwimmer models are given by the matter content and charges in table 3. We have taken the superpotential to be $W=\mathrm{Tr}\,X^3$, where X is the adjoint chiral superfield.

JHEP12(2002)047

	$\mathrm{SU}(N_c)$	$\mathrm{SU}(N_f)_Q$	$\mathrm{SU}(N_f)_{\widetilde{Q}}$	$\mathrm{U}(1)_R$
Q	$\square$	$\square$		$1-\frac{2N_c}{(k+1)N_f}$
$\widetilde{Q}$	$\overline{\square}$		$\square$	$1-\frac{2N_c}{(k+1)N_f}$
X	adj			$\frac{2}{k+1}$

Table 3: Matter content of Kutasov-Schwimmer models.

An orbifold preserving the $\mathcal{N}=1$ supersymmetry of those models is obtained by embedding the orbifold group in the gauge and global symmetries as for the SQCD case described above, and then the adjoint chiral multiplet decomposes into an adjoint under each of the $\mathrm{SU}(N)$ factors of the orbifold theory. By explicit calculation we find that for all N_f and N_c in the conformal region without accidental symmetry, $N_c < N_f < 2N_c$, the anomaly positivity conditions are satisfied.

These results provide evidence that orbifolds of theories with fixed points have fixed points themselves, also hinting at a correspondence between certain theories and their orbifolds, even at finite N. A more complete study would be useful. Comments on the ADS/CFT correspondence at finite N were made recently in [16].

6. Conclusions

We have demonstrated a simple relation between the gauge coupling functions of $\mathcal{N}=2$ $\mathrm{SU}(kN)$ pure gauge theory and $\mathcal{N}=1$ $\mathrm{SU}(N)^k$ gauge theory with bifundamental chiral multiplets. If the prescription given in [5] is generic for producing Seiberg-Witten curves of orbifolded field theories with a Coulomb branch, then this result is valid for all such theories. The problem of anomalous U(1)'s has not been satisfactorily understood in generic orbifolds of field theories. In the case studied above, the problem of anomalous U(1)'s is compensated for by axions in the twisted sectors of the orbifold theory [5]. In that case, the curves obtained are those of the orbifolded theory without the additional U(1)'s or twisted sector fields, as derived in [9]. It is not known to us whether this behavior is generic. This demonstrates a correspondence between one aspect of the orbifolded and parent theories at finite N. It is not know to us whether the Kahler potentials for the two theories behaves similarly.

We studied the anomaly positivity constraints [4] on theories with infrared fixed points and found that at large N these constraints are satisfied also in field theory orbifolds of such theories. At finite N we studied two classes of theories and their orbifolds and found no violation of the positivity constraints in the orbifolded theories. This provides some evidence that orbifolds of theories with conformal fixed points have conformal fixed points themselves, in keeping with the parent/orbifold correspondence.

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JHEP12(2002)047